Math - Introduction

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SET THEORY

DEFINITION: A set is a collection of objects which are called elements or members of the set

Examples:
Set of vegetables

$$V = \{\text{Aubergine}, \text{Courgette}, \text{Pepper}, \ldots\}$$

Set of AC⚡DC members

$$\lightning ACDC \lightning = \{\text{Bon Scott, Angus Young}, \ldots\}$$

However some sets cannot be listed a the ones previously shown, because they might have infinitely many elements. Consequently some times sets have to be defined according to a property.

Example:

The budget set: The consumer is constrained in the quantity of goods that can buy, according to the prices and her wealth, call:

- x_1: grams of cured ham with price p_{x_1}
- x_2: kilometres of public transport with price p_{x_2}
- Wealth: w euros

Then the set of bundles that the consumer can buy feasibly is:

$$\mathscr{B} = \left\{(x_1, x_2) \in \mathbb{R}^2 : p_{x_1} x_1 + p_{x_2} x_2 \leq w,\ x_1 \geq 0,\ x_2 \geq 0\right\}$$

NOTATION:

$$S = \{\text{Typical member: defining properties}\}$$

SET THEORY

SET MEMBERSHIP:

To indicate that an element x **belongs to a set** S, we write:

$$x \in S$$

And we write:

$$x \notin S$$

To indicate that it does not.

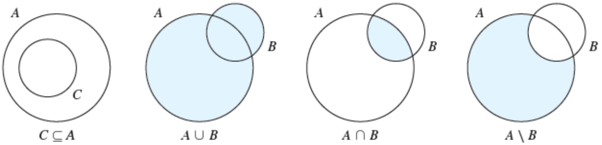
Also it might be interesting to note that a set Q is a **subset** of S

$$Q \subseteq S$$

SET THEORY

SET OPERATIONS:

- C subset of A: $C \subseteq A \iff x \in C \Rightarrow x \in A$
- A union B: $A \cup B = \{x : x \in A \text{ or } x \in B\}$
- A intersection B: $A \cap B = \{x : x \in A\ ,\ x \in B\}$
- A minus B: $A \setminus B = \{x : x \in A, x \notin B\}$



Exercise: Let $A = 2,3,4$, $B = 2,5,6$, $C = 5,6,2$ and $D = 1$

1. Determine which of the following statements are true: $4 \in C$; $5 \in C$; $A \subseteq B$; $D \subseteq C$; $B = C$; and $A = B$
2. Find $A \bigcup B$; $B \setminus A$; $A \bigcup B \bigcup C \bigcup D$; and $A \bigcap B \bigcap C \bigcap D$

Solutions:

1. False, True, False, False, True, False
2. $A \bigcup B = 2,3,4,5,6$; $B \setminus A = 5,6$; $A \bigcup B \bigcup C \bigcup D = 1,2,3,4,5,6$; and $A \bigcap B \bigcap C \bigcap D = \emptyset$

REAL NUMBERS

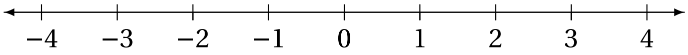
The most important set of all, at least for our purposes, is undoubtedly the real number line $\mathbb{R}$. But before jumping into its definition we need some basic notion of what numbers are.

The basic numbers are $1, 2, 3, \cdots$ **the Natural Numbers** ($\mathbb{N}$)

The natural numbers, together with 0 and their negative counterparts form the set of **Integers** ($\mathbb{Z}$), i.e. $\mathbb{N} \subseteq \mathbb{Z}$, which are

$$0, \pm 1, \pm 2, \pm 3, \ldots$$

And might be represented in the **number line**



After that come the **Rational Numbers** $\mathbb{Q}$ which are those that can be expressed as the quotient of two **Integers** $\frac{p}{q}$.

Notice that integers are a proper subset of rational numbers since any integer can be represented as a fraction of two integers, namely $\pm n = \pm \frac{n}{1}$.

Examples of these numbers are:

$$\frac{11}{70}, \quad 0 = \frac{0}{1}, \quad -19, \quad -\frac{10}{11}, \quad -1.23 = -\frac{123}{100}$$

REAL NUMBERS

However, we need to understand that rational numbers do not 'fill' the number line, there will be **holes**.

For example, $\sqrt{2}$ can not be represented as a fraction of two integers $\frac{p}{q}$

Proof: (by contradiction) let p, q be two co-prime numbers i.e. $gcd = 1$. Now, imagine there exists two integers such that $\sqrt{2} = \frac{p}{q}$, then

$$\text{Operating: } \sqrt{2} = \frac{p}{q}; \quad 2q^2 = p^2 \Rightarrow p = 2k \text{ for } k \in \mathbb{Z}$$

$$\text{Substituting: } 2q^2 = (2k)^2 \Rightarrow q^2 = 2k^2$$

If p, q are even numbers then they are not co-primes, hence a contradiction. This concludes that there is not such integers such that $\sqrt{2} = \frac{p}{q}$.

REAL NUMBERS

As seen rational numbers alone are not sufficient for filling the number line we need something else, the **Irrational numbers**.

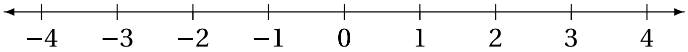
Irrational numbers are those decimals that are non-periodic such as $\sqrt{2}$, $-\sqrt{5}$, π, etc.

Since they are non-periodic, these numbers fill the holes left out by rational ones. In a certain way they ’close up’ the number line or the real line.

In this way the number line is said to be **dense**, because between two points there will always be rational and irrational numbers, and this can be done infinitely.

REAL NUMBERS

Since we have already constructed the real line



It will be useful to name subsets of this line, namely **Intervals**. The most commonly used are

Closed:	$\{x \in \mathbb{R} \vert a \leq x \leq b\}$	$x \in [a, b]$
Open:	$\{x \in \mathbb{R} \vert a < x < b\}$	$x \in (a, b)$
Right-semiclosed:	$\{x \in \mathbb{R} \vert a < x \leq b\}$	$x \in (a, b]$
Left-semiclosed:	$\{x \in \mathbb{R} \vert a \leq x < b\}$	$x \in [a, b)$
Unbounded:	$\{x \in \mathbb{R} \vert\ a \leq x < \infty\}$	$x \in [a, \infty)$

EQUATIONS

- **Variable**: it is an element representing an unknown value.
  Example: $x, y, z, \ldots$ are common ones
- **Expression**: it is a statement about of the value of something.
  Example: $x = 5$
- **Equation**: An equation is a statement that two expressions are equal.
  - **True equation**: when the statement of the equation is true
    Example: $7 = 3 + 4$
  - **False equation**: when the statement of the equation is false
    Example: $7 = 5$
  - **Algebraic equation**: if the equation involves dealing with a variable
    Example: $x + 4 = 7$ or $\frac{4}{x} = 2$

EQUATIONS

Usually we are interested in knowing for what values of the unknown variable the equation is a **true statement**.

The know this by isolating the variable to one side of the equation. The procedure usually involves to add, multiply, elevate, etc. to **both sides** in order to keep the equality being true.

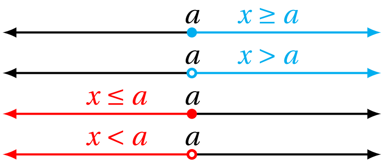
Practice yourself!! Solve the equation

$$\frac{z}{z-5}+\frac{1}{3}=\frac{-5}{z-5}$$

EQUATIONS

Inequality: it is a statement about if quantity is bigger(or smaller) than other.

- greater than or equal to:
- (strictly) greater than:
- less than or equal to:
- (strictly) less than:



Warning: when multiplying/dividing by a negative number the direction of inequality changes.

Exercise: are the following inequalities true?

$$(a)\ -6.15 > -7.16 \qquad (b)\ \frac{4}{5} > \frac{6}{7} \qquad (c)\ \frac{1}{2} - \frac{2}{3} < \frac{1}{4} - \frac{1}{3}$$

Solution: respectively true, false and true

Exercise: For what values of x is the inequality true? write it in set notation.

$$\frac{2x+1}{x-3} > 1$$

EQUATIONS

Solution: there are two cases

1. Both expressions positive at the same time

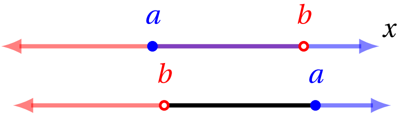
$$\left.\begin{array}{c}\left.\begin{array}{c} 2x+1>0 \iff x>-\frac{1}{2} \\ \text{and} \\ x-3>0 \iff x>3 \end{array}\right\} \Rightarrow \\ \\ \frac{2x+1}{x-3}>1 \iff \end{array} \begin{array}{c} x>3 \\ \\ \text{and} \\ x>-4 \end{array}\right\} \Rightarrow x>3$$

2. Both expressions negative. since $x-3<0$ then inequality changes

$$\frac{2x+1}{x-3}<1 \iff \left.\begin{array}{c} \left.\begin{array}{c} 2x+1<0 \iff x<-\frac{1}{2} \\ \text{and} \\ x-3<0 \iff x<3 \end{array}\right\} \Rightarrow \quad x<-\frac{1}{2} \\ \text{and} \\ x<-4 \end{array}\right\} \Rightarrow x<-4$$

And the final solution will be union of the two sets: $x>3 \cup x<-4$

Compound inequality: an inequality that combines two simple inequalities. OR and AND inequalities. They can have 1, infinitely many or no solution.

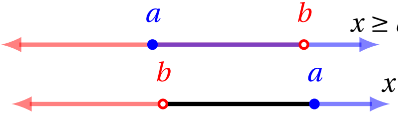


$x \geq a$ or $x < b = \mathbb{R}$

$x \geq a$ or $x < b$

Combine both with an OR or an AND.

Compound inequality: an inequality that combines two simple inequalities. OR and AND inequalities. They can have 1, infinitely many or no solution.



$x \geq a$ and $x < b = [a, b)$

$x \geq a$ and $x < b = \varnothing$

Notice that the second inequality is the empty set $\varnothing$, for there is no number smaller than b **AND** bigger than a at the same time.

SYSTEMS OF EQUATIONS

- **Systems of equations**: A System of Equations is when we have two or more equations working together. For the equations to "work together" they share one or more variables.
- **Equivalent systems**: systems with the same solutions.

Example:

$$ax + by = c$$
$$ax + dy = e$$

SYSTEM OF EQUATIONS

Calculation

- **Elimination:** Find a linear combination of the two equations and subtract one from the other to solve for the variable:

$$\begin{array}{r} ax + by = c \\ \underline{ax + dy = e} \\ (b - d)\, y = c - e \end{array}$$

- **Substitution:** solve for one variable in one equation and substitute the result into the other(s).

SYSTEM OF EQUATIONS

Calculation

Exercise: Solve the following system of equations

$$\begin{array}{rcrcr} 2x & + & 3y & = & 18 \\ 3x & - & 4y & = & -7 \end{array} \tag{1}$$

Solution:

Multiply the firs equation by 3 and the second by -2

$$\begin{array}{rcrcr} 6x & + & 9y & = & 54 \\ -6x & + & 8y & = & 14 \end{array}$$

Add up the two equations

$$17y = 68 \Rightarrow y = \frac{68}{17} \Rightarrow y = 4$$

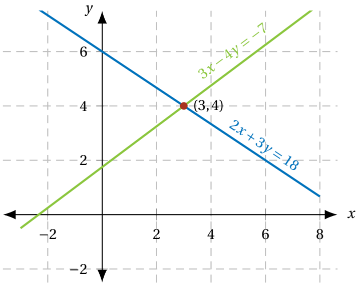
Substitute $y = 4$ in the first (or second) equation

$$2x + 12 = 18 \iff 2x = 6 \iff x = 3$$

SYSTEM OF EQUATIONS

Solutions

- **Consistent**: a system of equations is consistent if it has at least one solution (it can have infinite solutions).
  - **Independent**: a system of linear equations is independent if it has only one solution.

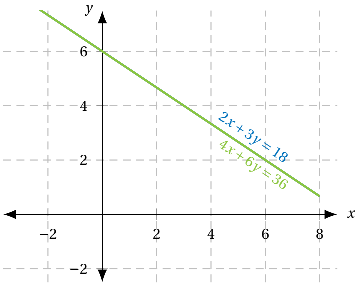


Refer to equation (1) of previous exercise

SYSTEM OF EQUATIONS

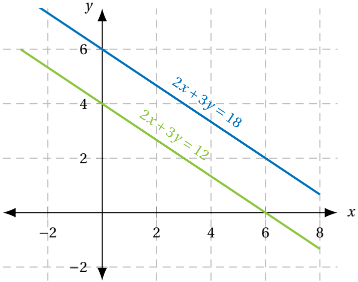
Solutions

- **Consistent**: a system of equations is consistent if it has at least one solution.
  - **Dependent**: a system of linear equations is dependent if it has infinitely many solutions.



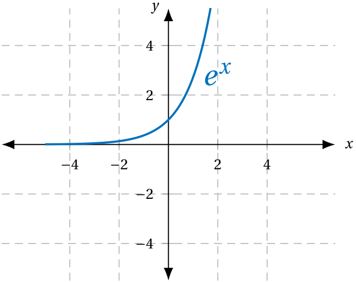
$$\begin{array}{rcrcr} 2x & + & 3y & = & 18 \\ 4x & + & 6y & = & 36 \end{array}$$

- **Inconsistent**: a system of equations is inconsistent if it has no solution.

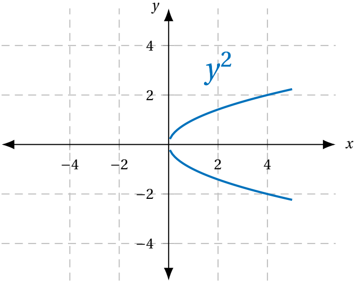


$$\begin{array}{ccccc} 2x & + & 3y & = & 18 \\ 2x & + & 3y & = & 12 \end{array}$$

Function: it is a special relationship where each input has a single unique output. It is often written as "$y = f(x)$" where x is the input value and y and $f(x)$ are the output.



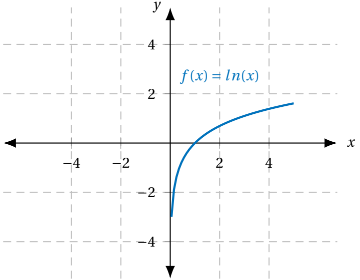
This is a function

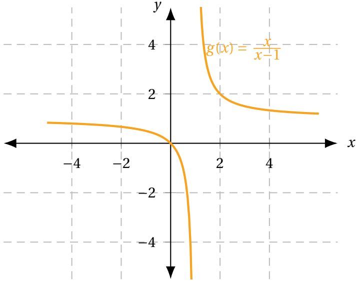


This is not a function

Domain: the set of numbers (inputs) for which the function has defined outputs.

Example:

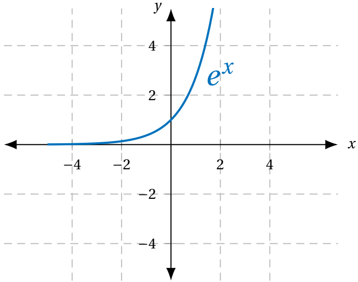




$$Dom\left(f(x)\right) = \{\forall x \in \mathbb{R} | x > 0\}$$

$$Dom\left(g(x)\right) = \{\forall x \in \mathbb{R} | x \in \mathbb{R} \setminus x = 1\}$$

Range: the set of actual values taken by the function
Example:



$$Rng\,(e^x) = \{\forall\, y \in \mathbb{R} | y > 0\}$$

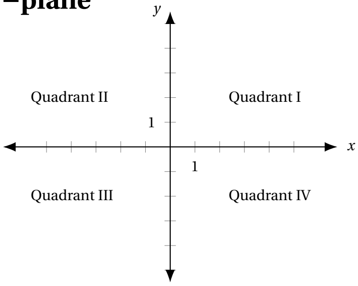
FUNCTIONS

BASICS

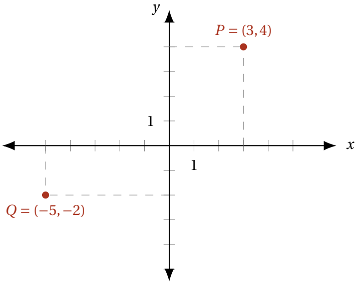
- **Co-domain**: the set of values that could possibly come out. The Co-domain is actually part of the definition of the function.
- **Range**: the set of actual values taken by the function

Example: for $x \in \mathbb{N}$ and $y = 2x$, the Co-domain is $Y = \left\{\forall y \in \mathbb{N} | y \in \mathbb{N}, \right\}$ whereas the range is $R = \left\{\forall y \in \mathbb{N} | y \text{ is even } \right\}$

The graph of a function f is the set of all points $(x, f(x))$, where x belongs to the domain of f. The graph is usually represented in the rectangular (or cartesian) **coordinate plane**, also called the **xy−plane**



A coordinated system

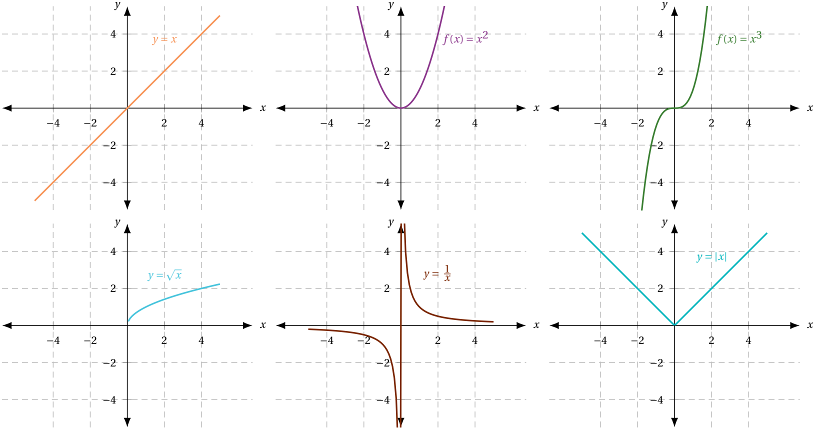


Points $(3,4)$ and $(-5,-2)$

Notice that $(3,4)$ and $(-5,-2)$ are the **coordinates** or **ordered pairs** of P and Q.

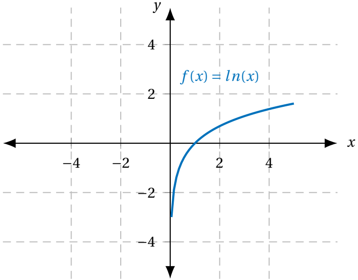
FUNCTIONS

GRAPHS

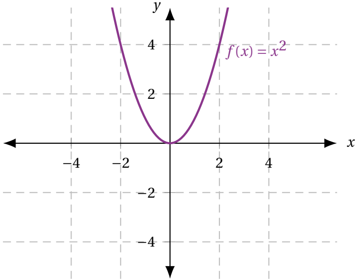


Useful functions

Injective (into): it means that every member of Y is matched by a unique member of X. So if $a \neq b$ then $f(a) \neq f(b)$.



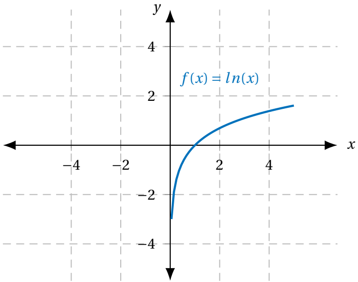
Injective



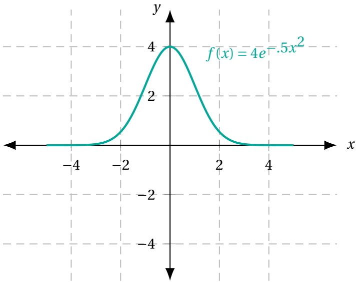
Not injective

NOTE: injectiveness is a property that a function has or does not, it is not a way of classifying functions.

Surjective (onto): means that every Y has at least one matching X (maybe more than one). $f(X) = Y$



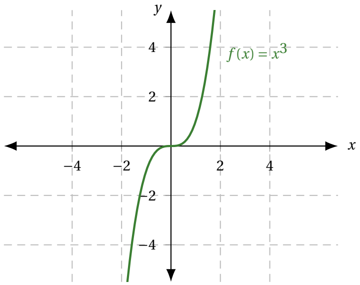
Surjective



Not surjective

NOTE: surjectiveness is a property that a function has or does not, it is not a way of classifying functions.

Bijective (One-to-one correspondence): Bijective means having both properties, Injective and Surjective, together. This property is important because it means that the **function is invertible**.



One-to-one Correspondence

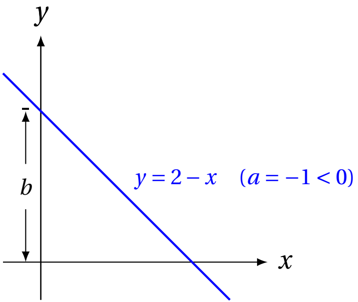
THE LINE

The **Linear function** occurs very often in economics. Is is defined as

$$y = ax + b, \quad a, b \in \mathbb{R}$$

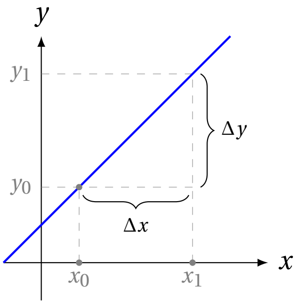
and the resulting graph is a straight line.

Example: as we can see below b tell us about the y-intercept whilst a is the slope of the function.



THE LINE

- **Slope**: it is a measure of the steepness of a line.



$$\text{Slope} = \frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x} = \frac{y_1 - y_0}{x_1 - x_0} = m$$

- **Point-Slope form:** $\left(y - b\right) = m\left(x - a\right),\ m, a, b \in \mathbb{R}$
- Slope-intercept form: $y = mx + b,\ m, c \in \mathbb{R}$.
- Standard form: $dx + ey = f,\ d, e, f \in \mathbb{R}$

THE LINE

Exercise: Construct the line that passes through this two points $(0,2)$ and $(4,3)$ and write in slope-intercept form.

Solution:
First you have to find the slope:

$$y_1 - y_0 = m(x_1 - x_0) \iff m = \frac{y_1 - y_0}{x_1 - x_0} \qquad \text{Substitute}$$

$$m = \frac{3-2}{4-0} = \frac{1}{4}$$

Now, since you have two points and the slope, use the point-slope form. I will use the point $(4,3)$ (you can use the other too) in

$$y - 3 = \frac{1}{4}(x-4)$$

Last, solve for y

$$y - 3 = \frac{1}{4}x - 1 \iff y = 2 + \frac{1}{4}x$$

QUADRATIC FUNCTION

Usually in economics linear relations, though useful, are not enough to explain the whole complexity of reality. Some times there are certain relation that as x increases the function $f(x)$ increases up to a point when it starts to decrease.

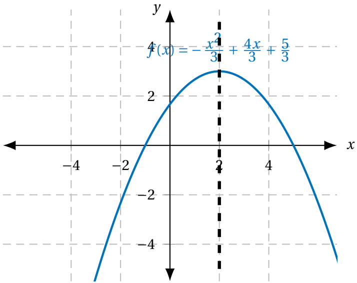
For this kind of behaviour we use the **Quadratic** function, whose general formula is

$$f(x) = ax^2 + bx + c$$

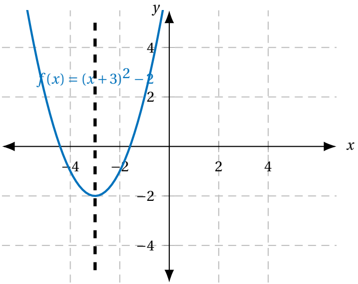
for $a \neq 0$, otherwise it would be a line.

QUADRATIC FUNCTION

The graph of the quadratic function is the **parabola**



$a < 0$



$a > 0$

Notice that the parabola is always symmetric about an axis of symmetry $x = \frac{-b}{2a}$

QUADRATIC FUNCTION

There are two important questions to solve about parabolas

1. For which values of x is $f(x) = ax^2 + bx + c = 0$
2. Where is the **vertex** of the parabola

For the first question we know that the formula to solve quadratic equations is

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Proof:

$$ax^2 + bx + c = 0 \iff x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a} = 0$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a} \iff \left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a} \iff x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

QUADRATIC FUNCTION

The most important bit of $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ is the determinant $b^2 - 4ac$, which not surprisingly determines if the function crosses the x-axis. There are tree cases:

1. $b^2 - 4ac > 0$ and the parabola will cross the x-axis twice
2. $b^2 - 4ac = 0$ and the parabola will just touch the x-coordinate
3. $b^2 - 4ac < 0$ and the parabola will not have solutions in the real line

Since the cross points are evenly far apart from each other, the vertex of the parabola will be place in the middle of them, namely $-\frac{b}{2a}$, and where the function will attain its minimum or maximum.

EXPONENCTIAL

- **Exponential Function:** it is a function of the form:

$$f(x) = b^{tx}$$

  - x: Exponent
  - t: Periods (complete)
  - b: Common ratio/base, is the rate of change of the function.

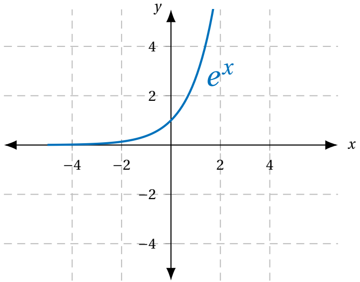
  waning: b cannot be negative

- **Exponential growth and decay:**
  - Exponential decay: $0 < b < 1$
  - Exponential growth: $b > 1$

EXPONENCTIAL

Graph:

- Asymptote: Exponential functions have a horizontal asymptote in 0, which can be moved up or down doing suitable transformations.
- Domain: $\mathbb{R}$
- Range: $\mathbb{R}_+ : \ (0, \infty)$



EXPONENCTIAL

The exponential function is specially valuable in order to model economic problems such as economic or population growth, calculate compound interest rates or work out the continuous depreciation of capital.

Exercise: a population like that of Zimbabwe is increasing at 3.5% annual rate, how much will the population have grown after 10 years? *Hint: use this formula:* $(1+r)^t = g$

Solution:

$$\Delta P_{ZW} = (1+0.035)^{10} \approx 1.41$$

Hence the population of Zimbabwe will be approximatelly 41% greater that it was at the begining of the period 10 year ago.

- **Logarithm Function:** it is a function of the form:

$$f(x) = log_b(x)$$

  - b: base
  - x: power/argument
- **Restrictions:**
  - $b > 0$: In an exponential function, the base b is always defined to be positive
  - $b \neq 1$: Assume $b = 1$, then $1^b = x$, which is not true for any value of $x \neq 1$
  - $x > 0$: Any positive number to any number is positive

Name	**Base**	**Regular Notation**	**Special Notation**
Common	10	$\log_{10}(x)$	$\log(x)$
Natural	e	$\log_e(x)$	$\ln(x)$

$$M = b^x \Leftrightarrow log_b M = x,\ N = b^y \Leftrightarrow log_b N = y \text{ and } a = b^O \Leftrightarrow log_b a = O$$

- Product Rule: $\log_b M \cdot N = \log_b M + \log_b N$.

$$\log_b(M \cdot N) = \log_b(b^x b^y) = \log_b(b^{x+y}) = x + y = \log_b M + \log_b N$$

- Quotient Rule: $\log_b\left(\frac{M}{N}\right) = \log_b M - \log_b N$

$$\log_b\left(\frac{M}{N}\right) = \log_b\left(\frac{b^x}{b^y}\right) = \log_b(b^{x-y}) = x - y = \log_b M - \log_b N$$

- Power Rule: $\log_b(M^p) = p \cdot \log_b M$

$$\log_b(M^p) = \log_b(\overbrace{M \cdot \ldots \cdot M}^{p \text{ times}}) = p \cdot \log_b M$$

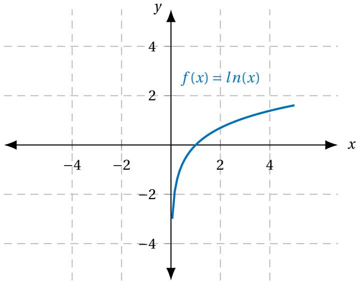
- Change of Base Rule: $\log_b(a) = \dfrac{\log_x(a)}{\log_x(b)}$

$$\log_b(a) = 0 \Longleftrightarrow b^0 = a \Longrightarrow \log_x b^0 = \log_x a \Longrightarrow 0 \log_x b = \log_x a \Longrightarrow 0 = \frac{\log_x(a)}{\log_x(b)}$$

LOGARITHM FUNCTION

GRAPH

- **Graph:**
  - Asymptote: Exponential functions have a vertical asymptote in 0, which can be moved left or right doing suitable transformations.
  - Domain: $\mathbb{R}_+ : (0, \infty)$
  - Range: $\mathbb{R}$



LOGARITHM FUNCTION

The main advantage brought to us by logarithms is that they allow to work with **ratios and exponential growth** in a very easy way. For example, look at the example given with the exponential functions

Exercise: a population like that of Zimbabwe is increasing at 3.5% annual rate, how long will the population take to double its size? *Hint: use the same formula:* $(1+r)^t = g$
Solution:

$$\Delta P_{ZW} = (1+0.035)^t = 2 \iff t\ln(1+0.035) = \ln 2$$
$$t = \frac{\ln 2}{\ln(1.035)} = 20.15 years$$

So, provided the population growth rate stays at 3.5% annually, Zimbabwe will see its population double in a span of 20.15 years.

LOGARITHM vs EXPONENTIAL

Relation logarithm vs exponential: One is the inverse function of the other

$$a = b^x \Leftrightarrow \log_b(a) = x$$

KA Video to understand properly how logarithms work.

FUNCTIONS

CHARACTERISTICS

- **Extrema:** maxima and minima considered collectively
- **Local maximum (minimum):** The height of the function at a point is greater (smaller) than the height anywhere else in that interval around a.
- **Global maximum (minimum):** The maximum or minimum over the entire function is called an "Absolute" or "Global" maximum or minimum

FUNCTIONS

CHARACTERISTICS

- Increasing (decreasing) functions: a function is increasing if y increases (decreases) as x increases.
- positive (negative) functions: a function is positive y it is above the horizontal axis, i.e. if $f(x)$ is positive (negative).

Average Rate of Change: it is the change of the function per unit of a variable the interval $x \in (a, b)$.

$$ARC = \frac{f(b) - f(a)}{b - a}$$

Combination of functions: As with addition, multiplication, power and exponent of numbers, there can be the same operations defined on functions:

- $\left(f+g\right)(x) = f(x) + g(x)$
- $\left(f-g\right)(x) = f(x) - g(x)$
- $\left(f\cdot g\right)(x) = f(x)\cdot g(x)$
- $\left(f/g\right)(x) = f(x)/g(x)$

Warning: the function $g(x)$ cannot be 0 when dividing

The domain of the new function will have the restrictions of **both** functions that made it.

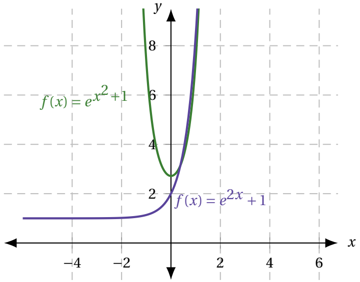
Examples: Addition, subtraction, multiplication and division

Composition of functions: It is applying one function to the results of another.

- $(g \circ f)(x) = g(f(x))$, first apply f(), then apply g()
- You must also respect the domain of the first function
- Some functions can be de-composed into two (or more) simpler functions.

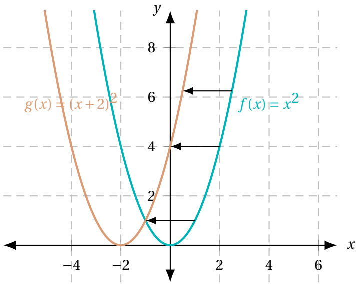
Example:

$$\left.\begin{aligned} f(x) &= e^{x} \\ g(x) &= x^2+1 \end{aligned}\right\} \Rightarrow \begin{aligned} f\circ g &= e^{x^2+1} \\ g\circ f &= e^{2x}+1 \end{aligned}$$

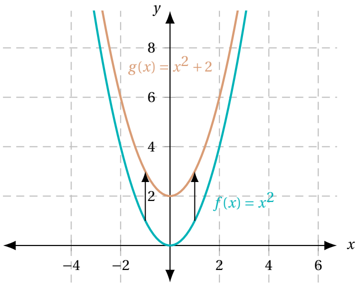


Shifting functions: A function can be shifted up/down or right/left adding/subtracting a number outside or inside the function respectively.

- Horizontally: $g(x) = f(x+a)$
- Vertically: $g(x) = f(x) + a$



Horizontally



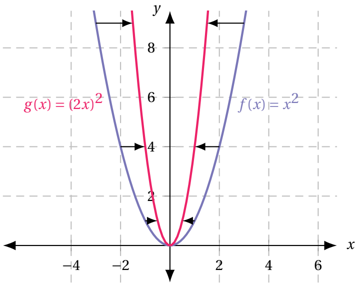
Vertically

FUNCTIONS

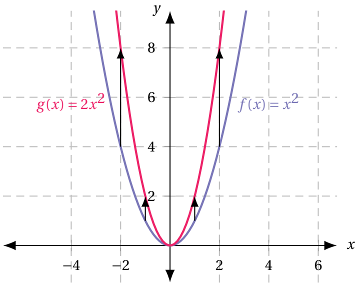
OPERATIONS

Stretching functions: A function can be stretched vertically or horizontally by multiplying/dividing a number outside or inside the function respectively.

- Horizontally: $g(x) = f(x \cdot a)$
- Vertically: $g(x) = f(x) \cdot a$



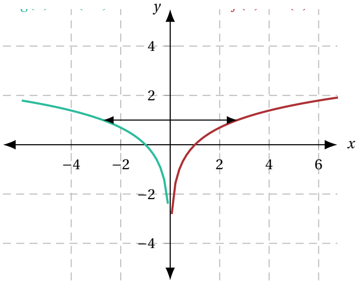
Horizontally



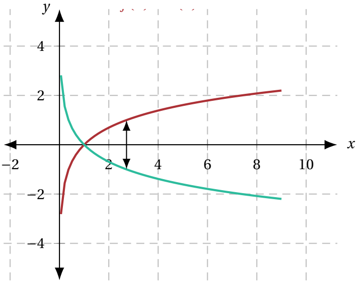
Vertically

Reflecting functions: a function can be flipped over the x/y axises by multiplying times -1 outside or inside the function respectively.

- y-axis: $g(x) = f(-x)$
- x-axis: $g(x) = -f(x)$



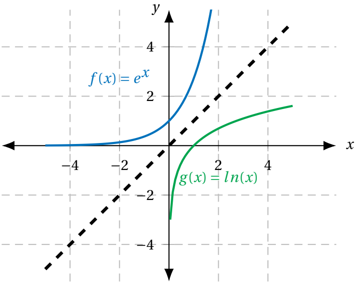
Horizontally



Vertically

Inverse functions: is a function that "reverses" another function: if the function f applied to an input x gives a result of y, then applying its inverse function g to y gives the result x, and vice versa. i.e., $f(x) = y \Leftrightarrow g(y) = x$

- Back to the original value: it gets to the point where we started $f\left(f(x)\right)^{-1} = x$
- Symmetry: the inverse function is symmetric across the $y = x$ line.



Invere functions

FUNCTIONS

INVERSES

- Solvability: Sometimes it is not possible to find the inverse of a function because the function cannot be solved for x.
- Invertibility: if each output has a unique input then the function is invertible. the horizontal line test.
- Not invertible: when for one value of y there exists two of x, the function is **not invertible**
- Domain: some functions do not have and inverse but it can be fixed restricting the domain.
- Notation: Inverse: $f^{-1}(x)$, reciprocal: $f(x)^{-1} = \frac{1}{f(x)}$, so $f^{-1}(x) \neq f(x)^{-1}$

Exercise: find the inverse of the following function $f(x) = \sqrt{3x+9}$.
Hint: solve for x, then swap the variables.

Solution: as pointed in the hint, first we have to solve for x

$$y = \sqrt{3x+9} \iff y^2 = 3x+9 \iff y^2 - 9 = 3x$$
$$\frac{1}{3}y^2 - 3 = x$$

And lastly swap variables

$$y = \frac{1}{3}x^2 - 3$$

SEQUENCE

BASICS

Sequence: it is an ordered list of numbers

- Terms: each number of the sequence
- Pattern: a rule that tells the following number of the list (if it exists)
- Function: sequences are functions with the caveat $n \in \mathbb{N}$
- Notation: $a(n) = a_n$

Arithmetic Sequence: a sequence in which the following number is the addition/subtraction of the previous one.

- Common difference: it is the constant difference between consecutive terms
- Recursive formula:

$$\begin{cases} a(1) = a_1 \\ a(n) = a(n-1) + b \end{cases}$$

- Explicit formula: $a(n) = a(1) + b(n-1)$

Exercise: find the first 10 terms and write the explicit formula of the following sequence

$$\begin{cases} a(1) = 5 \\ a(n) = a(n-1) + 3 \end{cases}$$

Solution: the first 10 terms are $5, 8, 11, 14, 17, 20, 23, 26, 29$ and 32 and the explicit formula

$$a(n) = 5 + 3(n-1)$$

Geometric Sequence: it is an array of numbers in which each term in the sequence is a fix multiple of the term before.

- Common ratio: The constant ratio between two different terms
- Recursive formula:

$$\begin{cases} a(1) = a_1 \\ a(n) = b \cdot a(n-1) \end{cases}$$

- Explicit formula: $a_n = a_1 \cdot b^{n-1}$

Exercise: find the first 5 terms and write the explicit formula of the following sequence

$$\begin{cases} a(1) = 5 \\ a(n) = 3 \cdot a(n-1) \end{cases}$$

Solution: the first 10 terms are $5, 15, 45, 135$ and 405 and the explicit formula

$$a(n) = 5 \cdot 3^{(n-1)}$$

SERIES

SUMMATION NOTATION

Economists usually make use of census data. Suppose a very far country is divided into 17 regions and the population of each region is denoted by N_i, then the total population is

$$N_1 + N_2 + N_3 + N_4 + N_5 + N_6 + N_7 + N_8 + N_9 + N_1 0 +$$
$$N_{11} + N_{12} + N_{13} + N_{14} + N_{15} + N_{16} + N_{17}$$

Often times is more useful to use the summation notation, thus the above sum can be written as

$$\sum_{i=1}^{17} N_i$$

Which reads as the sum of N_i from $i = 1$ to $i = 17$

Exercise: Compute $\sum_{k=3}^{6}(5k-3)$

Solution:

$$\sum_{k=3}^{6}(5k-3) = 5\cdot 3-3+5\cdot 4-3+5\cdot 5-3+5\cdot 6-3 =$$

$$= 5(3+4+5+6)-4\cdot 3 = 78$$

Exercise: express $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots$ in sum notation.

Solution:

$$\sum_{i=1}^{\infty}\frac{(-1)^{i-1}}{2n-1}$$

Series: it is the sum of a sequence
Arithmetic Series: it is the sum of terms of an arithmetic sequence.

$$S_n = \sum_{i=1}^{n} a_i = \frac{(a_1 + a_n)}{2} \cdot n$$

- Proof:

$$\begin{array}{rcccccccc} S_n = & a_1 & + & a_2 & + & \ldots & + & a_n & \\ S_n = & a_n & + & a_{n-1} & + & \ldots & + & a_1 & \\ \hline 2S_n = & (a_1 + a_n) & + & (a_1 + a_n) & + & \ldots & + & (a_1 + a_n) & \Rightarrow \end{array}$$

$$2S_n = \quad (a_1 + a_n) \cdot n \quad \Rightarrow \quad S_n = \frac{n \cdot (a_1 + a_n)}{2}$$

Geometric Series: it is the sum of the terms of a geometric sequence.

$$
\begin{aligned}
S_n &= \sum_{i=1}^{n} a_1 \cdot r^{n-1} \\
&= a_1 r^0 + a_1 r + a_1 r^2 + \dots + a_1 r^n \\
&= a_1 (1 + r + r^2 + \dots + r^n) \\
&= a_1 \cdot \frac{1 - r^n}{1 - r}, \text{ for } r < |1|
\end{aligned}
$$

Geometric Series: it is the sum of the terms of a geometric sequence.

- Proof: $1 + r + r^2 + ... + r^n = \frac{1}{1-r}$

Let:

$$\begin{array}{ccccccccccccl} 1 & + & r & + & r^2 & + & ... & + & r^n & & = S_n & & \text{Multiply by r} \\ & & r & + & ... & + & ... & + & r^n & +r^{n+1} & = rS_n & & \text{substract} \\ \hline 1 & & & & & & & & & -r^{n-1} & = (1-r)S_n & \Rightarrow & \end{array}$$

$$1 - r^{n-1} = (1-r)S_n \Rightarrow S_n = \frac{1-r^n}{1-r}$$

THE ZERO

- **Definition**
  - Number: ’zero’ is a number representing no amount
  - Placeholder: it also is a digit indicating there is no quantity. e.g.: $502 \neq 52$ here means there are no tens.
- **Characteristics:**
  - It is not negative nor positive
  - It is an even number. $0/2 = 0$ so there is no remainder
  - It is also an idea: if there is nothing to count how can we count it?

THE ZERO

PROPERTIES

Property	**Example**
$a \pm 0 = a$	$4 \pm 0 = 4$
$0 \times / : a = 0$	$0 \times / : 4 = 0$
$a/0 = undefined$	$4 \times / : 0 = undefined$
$0^a = 0 \; (a > 0)$	$0^4 = 0$
$0^0 = indeterminate$	$0^0 = indeterminate$
$0^a = undefined \; (a < 0)$	$0^{-4} = undefined$
$0! = 1$	$0! = 1$

https://www.mathsisfun.com/numbers/dividing-by-zero.html

INFINITY

- **Definition:** Infinity is something without an end.
- **Clarifications:**
  - Infinity does not grow, it does **not do** anything, it just **is**
  - It is not a **real number**
  - Infinity is just an **idea**
- **Using infinity:** Sometimes we can use infinity as if it were a real number, but it is not a real number. e.g.: $1+\infty=\infty$

INFINITY

Properties
$\infty + \infty = \infty$
$\infty \times \infty = \infty$
$x + \infty = \infty$
$x \times \infty = -\infty \ (x < 0)$

Undefined Operation
$0 \times \infty$
$\infty - \infty$
∞ / ∞
∞^0
1^∞